Investigating Quadratic Functions in Vertex Form

Focus on…

- identifying quadratic functions in vertex form
- determining the effect of \( a \), \( p \), and \( q \) on the graph of \( y = a(x - p)^2 + q \)
- analysing and graphing quadratic functions using transformations

The Bonneville Salt Flats is a large area in Utah, in the United States, that is a remnant of an ancient lake from glacial times. The surface is extremely flat, smooth, and hard, making it an ideal place for researchers, racing enthusiasts, and automakers to test high-speed vehicles in a safer manner than on a paved track. Recently, the salt flats have become the site of an annual time-trial event for alternative-fuel vehicles. At the 2007 event, one major automaker achieved a top speed of 335 km/h with a hydrogen-powered fuel-cell car, the highest-ever recorded land speed at the time for any fuel-cell-powered vehicle.

Suppose three vehicles are involved in speed tests. The first sits waiting at the start line in one test lane, while a second sits 200 m ahead in a second test lane. These two cars start accelerating constantly at the same time. The third car leaves 5 s later from the start line in a third lane.

The graph shows a function for the distance travelled from the start line for each of the three vehicles. How are the algebraic forms of these functions related to each other?
Part A: Compare the Graphs of \( f(x) = x^2 \) and \( f(x) = ax^2, a \neq 0 \)

1. a) Graph the following functions on the same set of coordinate axes, with or without technology.
   
   \[
   f(x) = x^2 \quad f(x) = -x^2 \\
   f(x) = 2x^2 \quad f(x) = -2x^2 \\
   f(x) = \frac{1}{2}x^2 \quad f(x) = -\frac{1}{2}x^2
   \]

   b) Describe how the graph of each function compares to the graph of \( f(x) = x^2 \), using terms such as narrower, wider, and reflection.

   c) What relationship do you observe between the parameter, \( a \), and the shape of the corresponding graph?

2. a) Using a variety of values of \( a \), write several of your own functions of the form \( f(x) = ax^2 \). Include both positive and negative values.

   b) Predict how the graphs of these functions will compare to the graph of \( f(x) = x^2 \). Test your prediction.

Reflect and Respond

3. Develop a rule that describes how the value of \( a \) in \( f(x) = ax^2 \) changes the graph of \( f(x) = x^2 \) when \( a \) is
   
   a) a positive number greater than 1
   b) a positive number less than 1
   c) a negative number

Part B: Compare the Graphs of \( f(x) = x^2 \) and \( f(x) = x^2 + q \)

4. a) Graph the following functions on the same set of coordinate axes, with or without technology.
   
   \[
   f(x) = x^2 \\
   f(x) = x^2 + 4 \\
   f(x) = x^2 - 3
   \]

   b) Describe how the graph of each function compares to the graph of \( f(x) = x^2 \).

   c) What relationship do you observe between the parameter, \( q \), and the location of the corresponding graph?

5. a) Using a variety of values of \( q \), write several of your own functions of the form \( f(x) = x^2 + q \). Include both positive and negative values.

   b) Predict how these functions will compare to \( f(x) = x^2 \). Test your prediction.

Reflect and Respond

6. Develop a rule that describes how the value of \( q \) in \( f(x) = x^2 + q \) changes the graph of \( f(x) = x^2 \) when \( q \) is
   
   a) a positive number 
   b) a negative number
Part C: Compare the Graphs of \( f(x) = x^2 \) and \( f(x) = (x - p)^2 \)

7. a) Graph the following functions on the same set of coordinate axes, with or without technology.
\[
\begin{align*}
  f(x) &= x^2 \\
  f(x) &= (x - 2)^2 \\
  f(x) &= (x + 1)^2
\end{align*}
\]

b) Describe how the graph of each function compares to the graph of \( f(x) = x^2 \).

c) What relationship do you observe between the parameter, \( p \), and the location of the corresponding graph?

8. a) Using a variety of values of \( p \), write several of your own functions of the form \( f(x) = (x - p)^2 \). Include both positive and negative values.

b) Predict how these functions will compare to \( f(x) = x^2 \). Test your prediction.

Reflect and Respond

9. Develop a rule that describes how the value of \( p \) in \( f(x) = (x - p)^2 \) changes the graph of \( f(x) = x^2 \) when \( p \) is

a) a positive number  

b) a negative number

The graph of a **quadratic function** is a **parabola**.

When the graph opens upward, the **vertex** is the lowest point on the graph. When the graph opens downward, the vertex is the highest point on the graph.
The y-coordinate of the vertex is called the **minimum value** if the parabola opens upward or the **maximum value** if the parabola opens downward.

The parabola is symmetric about a line called the **axis of symmetry**. This line divides the function graph into two parts so that the graph on one side is the mirror image of the graph on the other side. This means that if you know a point on one side of the parabola, you can determine a corresponding point on the other side based on the axis of symmetry.

The axis of symmetry intersects the parabola at the vertex.

The x-coordinate of the vertex corresponds to the equation of the axis of symmetry.

Quadratic functions written in **vertex form**, \( f(x) = a(x - p)^2 + q \), are useful when graphing the function. The vertex form tells you the location of the vertex \((p, q)\) as well as the shape of the parabola and the direction of the opening.

You can examine the parameters \(a, p,\) and \(q\) to determine information about the graph.
The Effect of Parameter $a$ in $f(x) = ax^2$ on the Graph of $f(x) = x^2$

Consider the graphs of the following functions:

\[
\begin{align*}
  f(x) &= x^2 \\
  f(x) &= 0.5x^2 \\
  f(x) &= -3x^2
\end{align*}
\]

- Parameter $a$ determines the orientation and shape of the parabola.
- The graph opens upward if $a > 0$ and downward if $a < 0$.
- If $-1 < a < 1$, the parabola is wider compared to the graph of $f(x) = x^2$.
- If $a > 1$ or $a < -1$, the parabola is narrower compared to the graph of $f(x) = x^2$.

The Effect of Parameter $q$ in $f(x) = x^2 + q$ on the Graph of $f(x) = x^2$

Consider the graphs of the following functions:

\[
\begin{align*}
  f(x) &= x^2 \\
  f(x) &= x^2 + 5 \\
  f(x) &= x^2 - 4
\end{align*}
\]

- Parameter $q$ translates the parabola vertically $q$ units relative to the graph of $f(x) = x^2$.
- The $y$-coordinate of the parabola's vertex is $q$. 

The parabola is wider in relation to the $y$-axis than $f(x) = x^2$ and opens upward.
The parabola is narrower in relation to the $y$-axis than $f(x) = x^2$ and opens downward.
The graph is translated 5 units up.
The graph is translated 4 units down.
The Effect of Parameter $p$ in $f(x) = (x - p)^2$ on the Graph of $f(x) = x^2$

Consider the graphs of the following functions:

- $f(x) = x^2$
- $f(x) = (x - 1)^2$  
  Since $p = +1$, the graph is translated 1 unit right.
- $f(x) = (x + 3)^2$  
  Since $p = -3$, the graph is translated 3 units left.

- Parameter $p$ translates the parabola horizontally $p$ units relative to the graph of $f(x) = x^2$.
- The $x$-coordinate of the parabola’s vertex is $p$.
- The equation of the axis of symmetry is $x - p = 0$ or $x = p$.

Combining Transformations

Consider the graphs of the following functions:

- $f(x) = x^2$
- $f(x) = -2(x - 3)^2 + 1$

- The parameter $a = -2$ determines that the parabola opens downward and is narrower than $f(x) = x^2$.
- The vertex of the parabola is located at $(3, 1)$ and represents a horizontal translation of 3 units right and a vertical translation of 1 unit up relative to the graph of $f(x) = x^2$.
- The equation of the axis of symmetry is $x - 3 = 0$ or $x = 3$.

In general:

- The sign of $a$ defines the direction of opening of the parabola. When $a > 0$, the graph opens upward, and when $a < 0$, the graph opens downward.
- The parameter $a$ also determines how wide or narrow the graph is compared to the graph of $f(x) = x^2$.
- The point $(p, q)$ defines the vertex of the parabola.
- The equation $x = p$ defines the axis of symmetry.
Example 1

Sketch Graphs of Quadratic Functions in Vertex Form

Determine the following characteristics for each function.
- the vertex
- the domain and range
- the direction of opening
- the equation of the axis of symmetry

Then, sketch each graph.

a) \( y = 2(x + 1)^2 - 3 \)  
   b) \( y = -\frac{1}{4}(x - 4)^2 + 1 \)

Solution

a) Use the values of \( a \), \( p \), and \( q \) to determine some characteristics of \( y = 2(x + 1)^2 - 3 \) and sketch the graph.

\[
y = 2(x + 1)^2 - 3
\]

\( a = 2 \), \( p = -1 \), \( q = -3 \)

Since \( p = -1 \) and \( q = -3 \), the vertex is located at \((-1, -3)\).

Since \( a > 0 \), the graph opens upward. Since \( a > 1 \), the parabola is narrower compared to the graph of \( y = x^2 \).

Since \( q = -3 \), the range is \( \{y \mid y \geq -3, y \in \mathbb{R}\} \).

The domain is \( \{x \mid x \in \mathbb{R}\} \).

Since \( p = -1 \), the equation of the axis of symmetry is \( x = -1 \).

Method 1: Sketch Using Transformations

Sketch the graph of \( y = 2(x + 1)^2 - 3 \) by transforming the graph of \( y = x^2 \).

- Use the points \((0, 0)\), \((1, 1)\), and \((-1, 1)\) to sketch the graph of \( y = x^2 \).
- Apply the change in width.

When using transformations to sketch the graph, you should deal with parameter \( a \) first, since its reference for wider or narrower is relative to the \( y \)-axis.

- Translate the graph.

How are \( p \) and \( q \) related to the direction of the translations and the location of the vertex?
**Method 2: Sketch Using Points and Symmetry**

- Plot the coordinates of the vertex, \((-1, -3)\), and draw the axis of symmetry, \(x = -1\).
- Determine the coordinates of one other point on the parabola.

The \(y\)-intercept is a good choice for another point.

Let \(x = 0\).

\[
y = 2(0 + 1)^2 - 3
\]
\[
y = 2(1)^2 - 3
\]
\[
y = -1
\]

The point is \((0, -1)\).

For any point other than the vertex, there is a corresponding point that is equidistant from the axis of symmetry. In this case, the corresponding point for \((0, -1)\) is \((-2, -1)\).

Plot these two additional points and complete the sketch of the parabola.

\[
y = 2(x + 1)^2 - 3
\]

\[
(-2, -1)
\]
\[
(-1, -3)
\]

\[
(0, -1)
\]

**b)** For the quadratic function \(y = -\frac{1}{4}(x - 4)^2 + 1\), \(a = -\frac{1}{4}\), \(p = 4\), and \(q = 1\).

The vertex is located at \((4, 1)\).

The graph opens downward and is wider than the graph \(y = x^2\).

The range is \(\{y \mid y \leq 1, y \in \mathbb{R}\}\).

The domain is \(\{x \mid x \in \mathbb{R}\}\).

The equation of the axis of symmetry is \(x = 4\).

Sketch the graph of \(y = -\frac{1}{4}(x - 4)^2 + 1\) by using the information from the vertex form of the function.
• Plot the vertex at (4, 1).
• Determine a point on the graph. For example, determine the $y$-intercept by substituting $x = 0$ into the function.

\[ y = -\frac{1}{4}(0 - 4)^2 + 1 \]
\[ y = -\frac{1}{4}(-4)^2 + 1 \]
\[ y = -4 + 1 \]
\[ y = -3 \]

The point $(0, -3)$ is on the graph.

For any point other than the vertex, there is a corresponding point that is equidistant from the axis of symmetry. In this case, the corresponding point of $(0, -3)$ is $(7, -3)$.

Plot these two additional points and complete the sketch of the parabola.

How are the values of $y$ affected when $a$ is $-\frac{1}{4}$?

How are $p$ and $q$ related to the direction of the translations and location of the vertex?

How is the shape of the curve related to the value of $a$?

**Your Turn**

Determine the following characteristics for each function.
• the vertex
• the domain and range
• the direction of opening
• the equations of the axis of symmetry

Then, sketch each graph.

a) \[ y = \frac{1}{2}(x - 2)^2 - 4 \]
b) \[ y = -3(x + 1)^2 + 3 \]
Determine a Quadratic Function in Vertex Form Given Its Graph

Determine a quadratic function in vertex form for each graph.

a)

\[
\begin{align*}
\text{Solution} \\
\text{a) Method 1: Use Points and Substitution} \\
\text{You can determine the equation of the function using the coordinates} \\
of the vertex and one other point. \\
The vertex is located at (5, -4), so \( p = 5 \) and \( q = -4 \). The graph opens upward, so the value of \( a \) is greater than 0. \\
Express the function as \\
f(x) = a(x - p)^2 + q \\
f(x) = a(x - 5)^2 + (-4) \\
f(x) = a(x - 5)^2 - 4 \\
\text{Choose one other point on the graph, such as (2, -1). Substitute the} \\
\text{values of} \ x \ \text{and} \ y \ \text{into the function and solve for} \ a. \\
f(x) = a(x - 5)^2 - 4 \\
\frac{-1}{4} = a(2 - 5)^2 - 4 \\
\frac{-1}{4} = a(-3)^2 - 4 \\
\frac{-1}{4} = a(9) - 4 \\
\frac{-1}{4} = 9a - 4 \\
3 = 9a \\
\frac{1}{3} = a \\
The quadratic function in vertex form is \( f(x) = \frac{1}{3}(x - 5)^2 - 4 \).
Method 2: Compare With the Graph of \( f(x) = x^2 \)

The vertex is located at \((5, -4)\), so \( p = 5 \) and \( q = -4 \). The graph involves a translation of 5 units to the right and 4 units down.

The graph opens upward, so the value of \( a \) is greater than 0.

To determine the value of \( a \), undo the translations and compare the vertical distances of points on the non-translated parabola relative to those on the graph of \( f(x) = x^2 \).

Since the vertical distances are one third as much, the value of \( a \) is \( \frac{1}{3} \).

The red graph of \( f(x) = \frac{1}{3}x^2 \) has been stretched vertically by a factor of \( \frac{1}{3} \) compared to the graph of \( f(x) = x^2 \).

Substitute the values \( a = \frac{1}{3} \), \( p = 5 \), and \( q = -4 \) into the vertex form, \( f(x) = a(x + p)^2 + q \).

The quadratic function in vertex form is \( f(x) = \frac{1}{3}(x - 5)^2 - 4 \).

b) You can determine the equation of the function using the coordinates of the vertex and one other point.

The vertex is located at \((0, 3)\), so \( p = 0 \) and \( q = 3 \). The graph opens downward, so the value of \( a \) is less than 0.

Express the function as
\[
\begin{align*}
f(x) &= a(x - p)^2 + q \\
f(x) &= a(x - 0)^2 + 3 \\
f(x) &= ax^2 + 3
\end{align*}
\]

Choose one other point on the graph, such as \((1, 1)\). Substitute the values of \( x \) and \( y \) into the function and solve for \( a \).

\[
\begin{align*}
f(x) &= ax^2 + 3 \\
1 &= a(1)^2 + 3 \\
1 &= a + 3 \\
-2 &= a
\end{align*}
\]

The quadratic function in vertex form is \( f(x) = -2x^2 + 3 \).
Your Turn

Determine a quadratic function in vertex form for each graph.

\[ f(x) = 0.8x^2 - 3 \]

\[ f(x) = 2(x - 1)^2 \]

\[ f(x) = -3(x + 2)^2 - 1 \]

**Example 3**

**Determine the Number of \(x\)-Intercepts Using \(a\) and \(q\)**

Determine the number of \(x\)-intercepts for each quadratic function.

a) \( f(x) = 0.8x^2 - 3 \)  

b) \( f(x) = 2(x - 1)^2 \)  

c) \( f(x) = -3(x + 2)^2 - 1 \)

**Solution**

You can determine the number of \(x\)-intercepts if you know the location of the vertex and direction of opening. Visualize the general position and shape of the graph based on the values of \(a\) and \(q\).

Determine the number of \(x\)-intercepts a quadratic function has by examining

- the value of \(a\) to determine if the graph opens upward or downward
- the value of \(q\) to determine if the vertex is above, below, or on the \(x\)-axis

a) \( f(x) = 0.8x^2 - 3 \)

<table>
<thead>
<tr>
<th>Value of (a)</th>
<th>Value of (q)</th>
<th>Visualize the Graph</th>
<th>Number of (x)-Intercepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a &gt; 0) \nthe graph \nopens upward\</td>
<td>(q &lt; 0) \nthe vertex \nis below the (x)-axis\</td>
<td>(f(x)) \n</td>
<td>2 \ncrosses the (x)-axis (twice, since it opens (upward\ from a\ vertex \below the (x)-axis)</td>
</tr>
</tbody>
</table>

b) \( f(x) = 2(x - 1)^2 \)

<table>
<thead>
<tr>
<th>Value of (a)</th>
<th>Value of (q)</th>
<th>Visualize the Graph</th>
<th>Number of (x)-Intercepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a &gt; 0) \nthe graph \nopens upward\</td>
<td>(q = 0) \nthe vertex \is on the (x)-axis\</td>
<td>(f(x)) \n</td>
<td>1 \ctouches the (x)-axis (once, since the vertex \is on the (x)-axis)</td>
</tr>
</tbody>
</table>

If you know that \(q\) is 0, does it matter what the value of \(a\) is? Where on the parabola is the \(x\)-intercept in this case?
c) \( f(x) = -3(x + 2)^2 - 1 \)

<table>
<thead>
<tr>
<th>Value of ( a )</th>
<th>Value of ( q )</th>
<th>Visualize the Graph</th>
<th>Number of ( x )-Intercepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &lt; 0 )</td>
<td>( q &lt; 0 )</td>
<td>the graph opens downward</td>
<td>the vertex is below the ( x )-axis</td>
</tr>
</tbody>
</table>

Your Turn

Determine the number of \( x \)-intercepts for each quadratic function without graphing.

a) \( f(x) = 0.5x^2 - 7 \)  
   b) \( f(x) = -2(x + 1)^2 \)  
   c) \( f(x) = -\frac{1}{6}(x - 5)^2 - 11 \)

Example 4

Model Problems Using Quadratic Functions in Vertex Form

The deck of the Lions’ Gate Bridge in Vancouver is suspended from two main cables attached to the tops of two supporting towers. Between the towers, the main cables take the shape of a parabola as they support the weight of the deck. The towers are 111 m tall relative to the water’s surface and are 472 m apart. The lowest point of the cables is approximately 67 m above the water’s surface.

a) Model the shape of the cables with a quadratic function in vertex form.

b) Determine the height above the surface of the water of a point on the cables that is 90 m horizontally from one of the towers. Express your answer to the nearest tenth of a metre.

Solution

a) Draw a diagram and label it with the given information.

Let the vertex of the parabolic shape be at the low point of the cables. Consider this point to be the origin.

Did You Know?

The Lions’ Gate Bridge carries over 60 000 vehicles per day on average. In 2009, the lights on the Lions’ Gate Bridge were replaced with a new LED lighting system. The change is expected to reduce the power consumption on the bridge by 90% and significantly cut down on maintenance.
Draw a set of axes. Let $x$ and $y$ represent the horizontal and vertical distances from the low point of the cables, respectively.

You can write a quadratic function if you know the coordinates of the vertex and one other point. The vertex is $(0, 0)$, since it is the origin. Determine the coordinates of the point at the top of each tower from the given distances.

Since the vertex is located at the origin, $(0, 0)$, no horizontal or vertical translation is necessary, and $p$ and $q$ are both zero. Therefore, the quadratic function is of the form $f(x) = ax^2$.

Substitute the coordinates of the top of one of the towers, $(236, 44)$, into the equation $f(x) = ax^2$ and solve for $a$.

\[
\begin{align*}
44 &= a(236)^2 \\
44 &= a(55,696) \\
44 &= 55,696a \\
\frac{44}{55,696} &= a \\
a &= \frac{11}{13,924}
\end{align*}
\]

$a$ is $\frac{11}{13,924}$ in lowest terms.

Represent the shape of the cables with the following quadratic function.

\[f(x) = \frac{11}{13,924}x^2\]

**b)** A point 90 m from one tower is $236 - 90$, or 146 m horizontally from the vertex. Substitute 146 for $x$ and determine the value of $f(146)$.

\[
\begin{align*}
f(x) &= \frac{11}{13,924}x^2 \\
f(146) &= \frac{11}{13,924}(146)^2 \\
&= \frac{11}{13,924}(21,316) \\
&= 16.839...
\end{align*}
\]

This is approximately 16.8 m above the low point in the cables, which are approximately 67 m above the water.

The height above the water is approximately $67 + 16.8$, or 83.8 m.
Your Turn
Suppose a parabolic archway has a width of 280 cm and a height of 216 cm at its highest point above the floor.

a) Write a quadratic function in vertex form that models the shape of this archway.
b) Determine the height of the archway at a point that is 50 cm from its outer edge.

Key Ideas

- For a quadratic function in vertex form, \( f(x) = a(x - p)^2 + q, a \neq 0 \), the graph:
  - has the shape of a parabola
  - has its vertex at \((p, q)\)
  - has an axis of symmetry with equation \( x = p \)
  - is congruent to \( f(x) = ax^2 \) translated horizontally by \( p \) units and vertically by \( q \) units

- Sketch the graph of \( f(x) = a(x - p)^2 + q \) by transforming the graph of \( f(x) = x^2 \).
  - The graph opens upward if \( a > 0 \).
  - If \( a < 0 \), the parabola is reflected in the \( x \)-axis; it opens downward.
  - If \(-1 < a < 1\), the parabola is wider compared to the graph of \( f(x) = x^2 \).
  - If \( a > 1 \) or \( a < -1 \), the parabola is narrower compared to the graph of \( f(x) = x^2 \).

- The parameter \( q \) determines the vertical position of the parabola.
  - If \( q > 0 \), then the graph is translated \( q \) units up.
  - If \( q < 0 \), then the graph is translated \( q \) units down.

- The parameter \( p \) determines the horizontal position of the parabola.
  - If \( p > 0 \), then the graph is translated \( p \) units to the right.
  - If \( p < 0 \), then the graph is translated \( p \) units to the left.

- You can determine a quadratic function in vertex form if you know the coordinates of the vertex and at least one other point.

- You can determine the number of \( x \)-intercepts of the graph of a quadratic function using the value of \( a \) to determine if the graph opens upward or downward and the value of \( q \) to determine if the vertex is above, below, or on the \( x \)-axis.
Check Your Understanding

Practise

1. Describe how you can obtain the graph of each function from the graph of \( f(x) = x^2 \). State the direction of opening, whether it has a maximum or a minimum value, and the range for each.
   a) \( f(x) = 7x^2 \)
   b) \( f(x) = \frac{1}{6}x^2 \)
   c) \( f(x) = -4x^2 \)
   d) \( f(x) = -0.2x^2 \)

2. Describe how the graphs of the functions in each pair are related. Then, sketch the graph of the second function in each pair, and determine the vertex, the equation of the axis of symmetry, the domain and range, and any intercepts.
   a) \( y = x^2 \) and \( y = x^2 + 1 \)
   b) \( y = x^2 \) and \( y = (x - 2)^2 \)
   c) \( y = x^2 \) and \( y = x^2 - 4 \)
   d) \( y = x^2 \) and \( y = (x + 3)^2 \)

3. Describe how to sketch the graph of each function using transformations.
   a) \( f(x) = (x + 5)^2 + 11 \)
   b) \( f(x) = -3x^2 - 10 \)
   c) \( f(x) = 5(x + 20)^2 - 21 \)
   d) \( f(x) = -\frac{1}{8}(x - 5.6)^2 + 13.8 \)

4. Sketch the graph of each function. Identify the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, the domain and range, and any intercepts.
   a) \( y = -(x - 3)^2 + 9 \)
   b) \( y = 0.25(x + 4)^2 + 1 \)
   c) \( y = -3(x - 1)^2 + 12 \)
   d) \( y = \frac{1}{2}(x - 2)^2 - 2 \)

5. a) Write a quadratic function in vertex form for each parabola in the graph.

   b) Suppose four new parabolas open downward instead of upward but have the same shape and vertex as each parabola in the graph. Write a quadratic function in vertex form for each new parabola.

   c) Write the quadratic functions in vertex form of four parabolas that are identical to the four in the graph but translated 4 units to the left.

   d) Suppose the four parabolas in the graph are translated 2 units down. Write a quadratic function in vertex form for each new parabola.

6. For the function \( f(x) = 5(x - 15)^2 - 100 \), explain how you can identify each of the following without graphing.
   a) the coordinates of the vertex
   b) the equation of the axis of symmetry
   c) the direction of opening
   d) whether the function has a maximum or minimum value, and what that value is
   e) the domain and range
   f) the number of \( x \)-intercepts
7. Without graphing, identify the location of the vertex and the axis of symmetry, the direction of opening and the maximum or minimum value, the domain and range, and the number of x-intercepts for each function.
   a) \( y = -4x^2 + 14 \)
   b) \( y = (x + 18)^2 - 8 \)
   c) \( y = 6(x - 7)^2 \)
   d) \( y = \frac{1}{9}(x + 4)^2 - 36 \)

8. Determine the quadratic function in vertex form for each parabola.
   a) 
   ![Graph](image-a)
   b) 
   ![Graph](image-b)
   c) 
   ![Graph](image-c)

9. Determine a quadratic function in vertex form that has the given characteristics.
   a) vertex at \((0, 0)\), passing through the point \((6, -9)\)
   b) vertex at \((0, -6)\), passing through the point \((3, 21)\)
   c) vertex at \((2, 5)\), passing through the point \((4, -11)\)
   d) vertex at \((-3, -10)\), passing through the point \((2, -5)\)

10. The point \((4, 16)\) is on the graph of \( f(x) = x^2 \). Describe what happens to the point when each of the following sets of transformations is performed in the order listed. Identify the corresponding point on the transformed graph.
    a) a horizontal translation of 5 units to the left and then a vertical translation of 8 units up
    b) a multiplication of the y-values by a factor of \(\frac{1}{4}\) and then a reflection in the x-axis
    c) a reflection in the x-axis and then a horizontal translation of 10 units to the right
    d) a multiplication of the y-values by a factor of 3 and then a vertical translation of 8 units down

11. Describe how to obtain the graph of \( y = 20 - 5x^2 \) using transformations on the graph of \( y = x^2 \).

12. Quadratic functions do not all have the same number of x-intercepts. Is the same true about y-intercepts? Explain.
13. A parabolic mirror was used to ignite the Olympic torch for the 2010 Winter Olympics in Vancouver and Whistler, British Columbia. Suppose its diameter is 60 cm and its depth is 30 cm.

a) Determine the quadratic function that represents its cross-sectional shape if the lowest point in the centre of the mirror is considered to be the origin, as shown.

b) How would the quadratic function be different if the outer edge of the mirror were considered the origin? Explain why there is a difference.

14. The finance team at an advertising company is using the quadratic function \( N(x) = -2.5(x - 36)^2 + 20000 \) to predict the effectiveness of a TV commercial for a certain product, where \( N \) is the predicted number of people who buy the product if the commercial is aired \( x \) times per week.

a) Explain how you could sketch the graph of the function, and identify its characteristics.

b) According to this model, what is the optimum number of times the commercial should be aired?

c) What is the maximum number of people that this model predicts will buy the product?

15. When two liquids that do not mix are put together in a container and rotated around a central axis, the surface created between them takes on a parabolic shape as they rotate. Suppose the diameter at the top of such a surface is 40 cm, and the maximum depth of the surface is 12 cm. Choose a location for the origin and write the function that models the cross-sectional shape of the surface.
16. The main section of the suspension bridge in Parc de la Gorge de Coaticook, Québec, has cables in the shape of a parabola. Suppose that the points on the tops of the towers where the cables are attached are 168 m apart and 24 m vertically above the minimum height of the cables.

a) Determine the quadratic function in vertex form that represents the shape of the cables. Identify the origin you used.

b) Choose two other locations for the origin. Write the corresponding quadratic function for the shape of the cables for each.

c) Use each quadratic function to determine the vertical height of the cables above the minimum at a point that is 35 m horizontally from one of the towers. Are your answers the same using each of your functions? Explain.

17. During a game of tennis, Natalie hits the tennis ball into the air along a parabolic trajectory. Her initial point of contact with the tennis ball is 1 m above the ground. The ball reaches a maximum height of 10 m before falling toward the ground. The ball is again 1 m above the ground when it is 22 m away from where she hit it. Write a quadratic function to represent the trajectory of the tennis ball if the origin is on the ground directly below the spot from which the ball was hit.

Did You Know?

Tennis originated from a twelfth-century French game called jeu de paume, meaning game of palm (of the hand). It was a court game where players hit the ball with their hands. Over time, gloves covered bare hands and, finally, racquets became the standard equipment. In 1873, Major Walter Wingfield invented a game called sphairstike (Greek for playing ball), from which modern outdoor tennis evolved.

18. Water is spraying from a nozzle in a fountain, forming a parabolic path as it travels through the air. The nozzle is 10 cm above the surface of the water. The water achieves a maximum height of 100 cm above the water’s surface and lands in the pool. The water spray is again 10 cm above the surface of the water when it is 120 cm horizontally from the nozzle. Write the quadratic function in vertex form to represent the path of the water if the origin is at the surface of the water directly below the nozzle.

Did You Know?

The suspension bridge in Parc de la Gorge de Coaticook in Québec claims to be the longest pedestrian suspension bridge in the world.
19. The function \( y = x^2 + 4 \) represents a translation of 4 units up, which is in the positive direction. The function \( y = (x + 4)^2 \) represents a translation of 4 units to the left, which is in the negative direction. How can you explain this difference?

20. In the movie, Apollo 13, starring Tom Hanks, scenes were filmed involving weightlessness. Weightlessness can be simulated using a plane to fly a special manoeuvre. The plane follows a specific inverted parabolic arc followed by an upward-facing recovery arc. Suppose the parabolic arc starts when the plane is at 7200 m and takes it up to 10 000 m and then back down to 7200 m again. It covers approximately 16 000 m of horizontal distance in total.

**a)** Determine the quadratic function that represents the shape of the parabolic path followed by the plane if the origin is at ground level directly below where the plane starts the parabolic arc.

**b)** Identify the domain and range in this situation.

Did You Know?

Passengers can experience the feeling of zero-g, or weightlessness, for approximately 30 s during each inverted parabolic manoeuvre made. During the recovery arc, passengers feel almost two-g, or almost twice the sensation of gravity. In addition to achieving weightlessness, planes such as these are also able to fly parabolic arcs designed to simulate the gravity on the moon (one sixth of Earth’s) or on Mars (one third of Earth’s).

21. Determine a quadratic function in vertex form given each set of characteristics. Explain your reasoning.

   **a)** vertex (6, 30) and a \( y \)-intercept of \(-24\)
   
   **b)** minimum value of \(-24\) and \(x\)-intercepts at \(-21\) and \(-5\)

Extend

22. **a)** Write quadratic functions in vertex form that represent three different trajectories the basketball shown can follow and pass directly through the hoop without hitting the backboard.

   **b)** Which of your three quadratic functions do you think represents the most realistic trajectory for an actual shot? Explain your thoughts.

   **c)** What do you think are a reasonable domain and range in this situation?

23. If the point \((m, n)\) is on the graph of \(f(x) = x^2\), determine expressions for the coordinates of the corresponding point on the graph of \(f(x) = a(x - p)^2 + q\).
Create Connections

24. **a)** Write a quadratic function that is related to \( f(x) = x^2 \) by a change in width, a reflection, a horizontal translation, and a vertical translation.

   **b)** Explain your personal strategy for accurately sketching the function.

25. Create your own specific examples of functions to explain how to determine the number of \( x \)-intercepts for quadratic functions of the form \( f(x) = a(x - p)^2 + q \) without graphing.

26. **MINI LAB** Graphing a function like \( y = -x^2 + 9 \) will produce a curve that extends indefinitely. If only a portion of the curve is desired, you can state the function with a restriction on the domain. For example, to draw only the portion of the graph of \( y = -x^2 + 9 \) between the points where \( x = -2 \) and \( x = 3 \), write \( y = -x^2 + 9, \{ x \mid -2 \leq x \leq 3, x \in \mathbb{R} \} \).

   **Materials**
   - 0.5-cm grid paper

   **Step 1** Use a piece of 0.5-cm grid paper. Draw axes vertically and horizontally through the centre of the grid. Label the axes with a scale.

   **Step 2** Plan out a line-art drawing that you can draw using portions of the graphs of quadratic and linear functions. As you create your illustration, keep a record of the functions you use. Add appropriate restrictions to the domain to indicate the portion of the graph you want.

   **Step 3** Use your records to make a detailed and accurate list of instructions/functions (including restrictions) that someone else could use to recreate your illustration.

   **Step 4** Trade your functions/instructions list with a partner. See if you can recreate each other’s illustration using only the list as a guide.

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**Project Corner**

**Parabolic Shape**

- Many suspension bridge cables, the arches of bridges, satellite dishes, reflectors in headlights and spotlights, and other physical objects often appear to have parabolic shape.

- You can try to model a possible quadratic relationship by drawing a set of axes on an image of a physical object that appears to be quadratic in nature, and using one or more points on the curve.

- What images or objects can you find that might be quadratic?