The Factor Theorem

Focus on . . .
- factoring polynomials
- explaining the relationship between the linear factors of a polynomial expression and the zeros of the corresponding function
- modelling and solving problems involving polynomial functions

Each year, more than 1 million intermodal containers pass through the Port of Vancouver. The total volume of these containers is over 2 million twenty-foot equivalent units (TEU). Suppose the volume, in cubic feet, of a 1-TEU container can be approximated by the polynomial function 

\[ V(x) = x^3 + 7x^2 - 28x + 20, \]

where \( x \) is a positive real number. What dimensions, in terms of \( x \), could the container have?

Did You Know?
An intermodal container is a standard-sized metal box that can be easily transferred between different modes of transportation, such as ships, trains, and trucks. A TEU represents the volume of a 20-ft intermodal container. Although container heights vary, the equivalent of 1 TEU is accepted as 1360 ft³.

Investigate Determining the Factors of a Polynomial

A: Remainder for a Factor of a Polynomial

1. a) Determine the remainder when \( x^3 + 2x^2 - 5x - 6 \) is divided by \( x + 1 \).
   b) Determine the quotient \( \frac{x^3 + 2x^2 - 5x - 6}{x + 1} \). Write the corresponding statement that can be used to check the division.
   c) Factor the quadratic portion of the statement written in part b).
   d) Write \( x^3 + 2x^2 - 5x - 6 \) as the product of its three factors.
   e) What do you notice about the remainder when you divide \( x^3 + 2x^2 - 5x - 6 \) by any one of its three factors?

Reflect and Respond

2. What is the relationship between the remainder and the factors of a polynomial?

B: Determine Factors

3. Which of the following are factors of \( P(x) = x^3 - 7x + 6 \)?
   Justify your reasoning.
   a) \( x + 1 \)  b) \( x - 1 \)  c) \( x + 2 \)
   d) \( x - 2 \)  e) \( x + 3 \)  f) \( x - 3 \)

Why would a factor such as \( x - 5 \) not be considered as a possible factor?
Reflect and Respond

4. Write a statement that describes the condition when a divisor \( x - a \) is a factor of a polynomial \( P(x) \).

5. What are the relationships between the factors of a polynomial expression, the zeros of the corresponding polynomial function, the \( x \)-intercepts of the graph of the corresponding polynomial function, and the remainder theorem?

6. a) Describe a method you could use to determine the factors of a polynomial.
   b) Use your method to determine the factors of \( f(x) = x^3 + 2x^2 - x - 2 \).
   c) Verify your answer.

The factor theorem states that \( x - a \) is a factor of a polynomial in \( x \), \( P(x) \), if and only if \( P(a) = 0 \).

For example, given the polynomial \( P(x) = x^3 - x^2 - 5x + 2 \), determine if \( x - 1 \) and \( x + 2 \) are factors by calculating \( P(1) \) and \( P(-2) \), respectively.

\[
\begin{align*}
P(x) &= x^3 - x^2 - 5x + 2 \\
P(1) &= 1^3 - 1^2 - 5(1) + 2 \\
P(-2) &= (-2)^3 - (-2)^2 - 5(-2) + 2 \\
P(1) &= -3 \\
P(-2) &= 10 + 2 \\
P(1) &= -3 \\
P(-2) &= 0
\end{align*}
\]

Since \( P(1) = -3 \), \( P(x) \) is not divisible by \( x - 1 \). Therefore, \( x - 1 \) is not a factor of \( P(x) \).

Since \( P(-2) = 0 \), \( P(x) \) is divisible by \( x + 2 \). Therefore, \( x + 2 \) is a factor of \( P(x) \).

The zeros of a polynomial function are related to the factors of the polynomial. The graph of \( P(x) = x^3 - x^2 - 4x + 4 \) shows that the zeros of the function, or the \( x \)-intercepts of the graph, are at \( x = -2 \), \( x = 1 \), and \( x = 2 \). The corresponding factors of the polynomial are \( x + 2 \), \( x - 1 \), and \( x - 2 \).
Example 1

Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial \( P(x) = x^3 - 3x^2 - x + 3 \)? Justify your answers.

a) \( x - 1 \)
b) \( x + 1 \)
c) \( x + 3 \)
d) \( x - 3 \)

Solution

a) Use the factor theorem to evaluate \( P(a) \) given \( x - a \).
   For \( x - 1 \), substitute \( x = 1 \) into the polynomial expression.
   \[ P(x) = x^3 - 3x^2 - x + 3 \]
   \[ P(1) = 1^3 - 3(1)^2 - 1 + 3 \]
   \[ P(1) = 1 - 3 - 1 + 3 \]
   \[ P(1) = 0 \]
   Since the remainder is zero, \( x - 1 \) is a factor of \( P(x) \).

b) For \( x + 1 \), substitute \( x = -1 \) into the polynomial expression.
   \[ P(x) = x^3 - 3x^2 - x + 3 \]
   \[ P(-1) = (-1)^3 - 3(-1)^2 - (-1) + 3 \]
   \[ P(-1) = -1 - 3 + 1 + 3 \]
   \[ P(-1) = 0 \]
   Since the remainder is zero, \( x + 1 \) is a factor of \( P(x) \).

c) For \( x + 3 \), substitute \( x = -3 \) into the polynomial expression.
   \[ P(x) = x^3 - 3x^2 - x + 3 \]
   \[ P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 3 \]
   \[ P(-3) = -27 - 27 + 3 + 3 \]
   \[ P(-3) = -48 \]
   Since the remainder is not zero, \( x + 3 \) is not a factor of \( P(x) \).

d) For \( x - 3 \), substitute \( x = 3 \) into the polynomial expression.
   \[ P(x) = x^3 - 3x^2 - x + 3 \]
   \[ P(3) = 3^3 - 3(3)^2 - 3 + 3 \]
   \[ P(3) = 27 - 27 - 3 + 3 \]
   \[ P(3) = 0 \]
   Since the remainder is zero, \( x - 3 \) is a factor of \( P(x) \).

Your Turn

Determine which of the following binomials are factors of the polynomial \( P(x) = x^3 + 2x^2 - 5x - 6 \).
\( x - 1, x + 1, x - 2, x + 2, x - 3, x + 3, x - 6, x + 6 \)
**Possible Factors of a Polynomial**

When factoring a polynomial, $P(x)$, it is helpful to know which integer values of $a$ to try when determining if $P(a) = 0$.

Consider the polynomial $P(x) = x^3 - 7x^2 + 14x - 8$. If $x = a$ satisfies $P(a) = 0$, then $a^3 - 7a^2 + 14a - 8 = 0$, or $a^3 - 7a^2 + 14a = 8$. Factoring out the common factor on the left side of the equation gives the product $a(a^2 - 7a + 14) = 8$. Then, the possible integer values for the factors in the product on the left side are the factors of 8. They are $±1$, $±2$, $±4$, and $±8$.

The relationship between the factors of a polynomial and the constant term of the polynomial is stated in the **integral zero theorem**.

The integral zero theorem states that if $x - a$ is a factor of a polynomial function $P(x)$ with integral coefficients, then $a$ is a factor of the constant term of $P(x)$.

**Example 2**

**Factor Using the Integral Zero Theorem**

**a)** Factor $2x^3 - 5x^2 - 4x + 3$ fully.

**b)** Describe how to use the factors of the polynomial expression to determine the zeros of the corresponding polynomial function.

**Solution**

**a)** Let $P(x) = 2x^3 - 5x^2 - 4x + 3$. Find a factor by evaluating $P(x)$ for values of $x$ that are factors of 3: $±1$ and $±3$.

Test the values.

$P(x) = 2x^3 - 5x^2 - 4x + 3$

$P(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3$

$P(1) = 2 - 5 - 4 + 3$

$P(1) = -4$

Since $P(1) ≠ 0$, $x - 1$ is not a factor of $2x^3 - 5x^2 - 4x + 3$.

$P(x) = 2x^3 - 5x^2 - 4x + 3$

$P(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$

$P(-1) = -2 - 5 + 4 + 3$

$P(-1) = 0$

Since $P(-1) = 0$, $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$.

Use synthetic or long division to find the other factors.

\[
\begin{array}{cccc|c}
+1 & 2 & -5 & -4 & 3 \\
- & & & & \\
\hline x & 2 & -7 & 3 & 0 \\
\end{array}
\]

The remaining factor is $2x^2 - 7x + 3$.

So, $2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$.

Factoring $2x^2 - 7x + 3$ gives $(2x - 1)(x - 3)$.

Therefore, $2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x - 1)(x - 3)$.
Since the factors of \(2x^3 - 5x^2 - 4x + 3\) are \(x + 1\), \(2x - 1\), and \(x - 3\), the corresponding zeros of the function are \(-1\), \(\frac{1}{2}\), and \(3\). Confirm the zeros by graphing \(P(x)\) and using the trace or zero feature of a graphing calculator.

**Your Turn**

What is the factored form of \(x^3 - 4x^2 - 11x + 30\)? How can you use the graph of the corresponding polynomial function to simplify your search for integral roots?

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**Example 3**

**Factor Higher-Degree Polynomials**

Fully factor \(x^4 - 5x^3 + 2x^2 + 20x - 24\).

**Solution**

Let \(P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24\).

Find a factor by testing factors of \(-24\): \(±1, ±2, ±3, ±4, ±6, ±8, ±12, \) and \(±24\)

\[
P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24
\]
\[
P(1) = 1^4 - 5(1)^3 + 2(1)^2 + 20(1) - 24
\]
\[
P(1) = 1 - 5 + 2 + 20 - 24
\]
\[
P(1) = -6
\]

\[
P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24
\]
\[
P(-1) = (-1)^4 - 5(-1)^3 + 2(-1)^2 + 20(-1) - 24
\]
\[
P(-1) = 1 + 5 + 2 - 20 - 24
\]
\[
P(-1) = -36
\]

\[
P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24
\]
\[
P(2) = 2^4 - 5(2)^3 + 2(2)^2 + 20(2) - 24
\]
\[
P(2) = 16 - 40 + 8 + 40 - 24
\]
\[
P(2) = 0
\]

Since \(P(2) = 0\), \(x - 2\) is a factor of \(x^4 - 5x^3 + 2x^2 + 20x - 24\).

Use division to find the other factors.

\[
\begin{array}{c|cccc}
-2 & 1 & -5 & 2 & 20 \\
 & & -2 & 6 & 8 \\
\hline
x & 1 & -3 & -4 & 12 \\
\end{array}
\]

When should you stop testing possible factors?
The remaining factor is \( x^3 - 3x^2 - 4x + 12 \).

**Method 1: Apply the Factor Theorem Again**

Let \( f(x) = x^3 - 3x^2 - 4x + 12 \).

Since \( f(2) = 0 \), \( x - 2 \) is a second factor.

Use division to determine that the other factor is \( x^2 - x - 6 \).

\[
\begin{array}{c|cccc}
-2 & 1 & -3 & -4 & 12 \\
- & -2 & 2 & 12 \\
\hline
\ & 1 & -1 & -6 & 0
\end{array}
\]

Factoring \( x^2 - x - 6 \) gives \( (x + 2)(x - 3) \).

Therefore,

\[
x^4 - 5x^3 + 2x^2 + 20x - 24 = (x - 2)(x - 2)(x + 2)(x - 3)
\]

\[
= (x - 2)^2(x + 2)(x - 3)
\]

**Method 2: Factor by Grouping**

\[
x^3 - 3x^2 - 4x + 12 = x^2(x - 3) - 4(x - 3)
\]

Group the first two terms and factor out \( x^2 \). Then, group the second two terms and factor out \(-4\).

\[
= (x - 3)(x^2 - 4)
\]

Factor out \( x - 3 \).

\[
= (x - 3)(x - 2)(x + 2)
\]

Factor the difference of squares \( x^2 - 4 \).

Therefore,

\[
x^4 - 5x^3 + 2x^2 + 20x - 24 = (x - 2)(x - 3)(x - 2)(x + 2)
\]

\[
= (x - 2)^2(x + 2)(x - 3)
\]

**Your Turn**

What is the fully factored form of \( x^4 - 3x^3 - 7x^2 + 15x + 18 \)?

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**Example 4**

**Solve Problems Involving Polynomial Expressions**

An intermodal container that has the shape of a rectangular prism has a volume, in cubic feet, represented by the polynomial function

\[
V(x) = x^3 + 7x^2 - 28x + 20,
\]

where \( x \) is a positive real number.

What are the factors that represent possible dimensions, in terms of \( x \), of the container?
**Solution**

**Method 1: Use Factoring**

The possible integral factors correspond to the factors of the constant term of the polynomial, 20: \(±1, ±2, ±4, ±5, ±10, \) and \(±20\). Use the factor theorem to determine which of these values correspond to the factors of the polynomial. Use a graphing calculator or spreadsheet to help with the multiple calculations.

<table>
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<th>(x)</th>
<th>(P(x))</th>
</tr>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>-1</td>
<td>54</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
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<td>4</td>
<td>84</td>
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<td>-4</td>
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<tr>
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<tr>
<td>20</td>
<td>10260</td>
</tr>
<tr>
<td>-20</td>
<td>-4620</td>
</tr>
</tbody>
</table>

The values of \(x\) that result in a remainder of zero are \(-10, 1, \) and \(2\). The factors that correspond to these values are \(x + 10, x - 1, \) and \(x - 2\). The factors represent the possible dimensions, in terms of \(x\), of the container.

**Method 2: Use Graphing**

Since the zeros of the polynomial function correspond to the factors of the polynomial expression, use the graph of the function to determine the factors.

The trace or zero feature of a graphing calculator shows that the zeros of the function are \(x = -10, x = 1, \) and \(x = 2\). These correspond to the factors \(x + 10, x - 1, \) and \(x - 2\). The factors represent the possible dimensions, in terms of \(x\), of the container.

**Your Turn**

A form that is used to make large rectangular blocks of ice comes in different dimensions such that the volume, \(V\), in cubic centimetres, of each block can be modelled by \(V(x) = x^3 + 7x^2 - 28x + 20\), where \(x\) is in centimetres. Determine the possible dimensions, in terms of \(x\), that result in this volume.
Practise

1. What is the corresponding binomial factor of a polynomial, \( P(x) \), given the value of the zero?
   a) \( P(1) = 0 \)
   b) \( P(-3) = 0 \)
   c) \( P(4) = 0 \)
   d) \( P(a) = 0 \)

2. Determine whether \( x - 1 \) is a factor of each polynomial.
   a) \( x^3 - 3x^2 + 4x - 2 \)
   b) \( 2x^3 - x^2 - 3x - 2 \)
   c) \( 3x^3 - x - 3 \)
   d) \( 2x^3 + 4x^2 - 5x - 1 \)
   e) \( x^4 - 3x^3 + 2x^2 - x + 1 \)
   f) \( 4x^4 - 2x^3 + 3x^2 - 2x + 1 \)

3. State whether each polynomial has \( x + 2 \) as a factor.
   a) \( 5x^2 + 2x + 6 \)
   b) \( 2x^3 - x^2 - 5x - 8 \)
   c) \( 2x^3 + 2x^2 - x - 6 \)
   d) \( x^4 - 2x^2 + 3x - 4 \)
   e) \( x^4 + 3x^3 - x^2 - 3x + 6 \)
   f) \( 3x^4 + 5x^3 + x - 2 \)

4. What are the possible integral zeros of each polynomial?
   a) \( P(x) = x^3 + 3x^2 - 6x - 8 \)
   b) \( P(s) = s^3 + 4s^2 - 15s - 18 \)
   c) \( P(n) = n^3 - 3n^2 - 10n + 24 \)
   d) \( P(p) = p^4 - 2p^3 - 8p^2 + 3p - 4 \)
   e) \( P(z) = z^4 + 5z^3 + 2z^2 + 7z - 15 \)
   f) \( P(y) = y^4 - 5y^3 - 7y^2 + 21y + 4 \)
5. Factor fully.
   a) \( P(x) = x^3 - 6x^2 + 11x - 6 \)
   b) \( P(x) = x^3 + 2x^2 - x - 2 \)
   c) \( P(v) = v^3 + v^2 - 16v - 16 \)
   d) \( P(x) = x^4 + 4x^3 - 7x^2 - 34x - 24 \)
   e) \( P(k) = k^3 + 3k^2 - 5k^2 - 15k^2 + 4k + 12 \)

6. Factor fully.
   a) \( x^3 - 2x^2 - 9x + 18 \)
   b) \( t^3 + t^2 - 22t - 40 \)
   c) \( h^3 - 27h + 10 \)
   d) \( x^5 + 8x^3 + 2x - 15 \)
   e) \( q^4 + 2q^3 + 2q^2 - 2q - 3 \)

Apply

7. Determine the value(s) of \( k \) so that the binomial is a factor of the polynomial.
   a) \( x^2 - x + k, x - 2 \)
   b) \( x^2 - 6x - 7, x + k \)
   c) \( x^3 + 4x^2 + x + k, x + 2 \)
   d) \( x^2 + kx - 16, x - 2 \)

8. The volume, \( V(h) \), of a bookcase can be represented by the expression \( h^3 - 2h^2 + h \), where \( h \) is the height of the bookcase. What are the possible dimensions of the bookcase in terms of \( h \)?

9. A racquetball court has a volume that can be represented by the polynomial \( V(l) = l^3 - 2l^2 - 15l \), where \( l \) is the length of the side walls. Factor the expression to determine the possible width and height of the court in terms of \( l \).

10. Mikisiti Sails (1939–2008), an Inuit artist from Cape Dorset, Nunavut, was the son of famous soapstone carver Pauta Sails. Mikisita’s preferred theme was wildlife presented in a minimal but graceful and elegant style. Suppose a carving is created from a rectangular block of soapstone whose volume, \( V \), in cubic centimetres, can be modelled by \( V(x) = x^3 + 5x^2 - 2x - 24 \). What are the possible dimensions of the block, in centimetres, in terms of binomials of \( x \)?

11. The volume of water in a rectangular fish tank can be modelled by the polynomial \( V(x) = x^3 + 14x^2 + 63x + 90 \). If the depth of the tank is given by the polynomial \( x + 6 \), what polynomials represent the possible length and width of the fish tank?
12. When a certain type of plastic is cut into sections, the length of each section determines its relative strength. The function \( f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100 \) describes the relative strength of a section of length \( x \) feet. After testing the plastic, engineers discovered that 5-ft sections were extremely weak.

a) Why is \( x - 5 \) a possible factor when \( x = 5 \) is the length of the pipe? Show that \( x - 5 \) is a factor of the polynomial function.

b) Are there other lengths of plastic that are extremely weak? Explain your reasoning.

13. The product of four integers is \( x^4 + 6x^3 + 11x^2 + 6x \), where \( x \) is one of the integers. What are possible expressions for the other three integers?

14. Consider the polynomial \( f(x) = ax^4 + bx^3 + cx^2 + dx + e \), where \( a + b + c + d + e = 0 \). Show that this polynomial is divisible by \( x - 1 \).

15. Determine the values of \( m \) and \( n \) so that the polynomials \( 2x^3 + mx^2 + nx - 3 \) and \( x^3 - 3mx^2 + 2nx + 4 \) are both divisible by \( x - 2 \).

16. a) Factor each polynomial.
   i) \( x^3 - 1 \)
   ii) \( x^3 - 27 \)
   iii) \( x^3 + 1 \)
   iv) \( x^3 + 64 \)

b) Use the results from part a) to decide whether \( x + y \) or \( x - y \) is a factor of \( x^3 + y^3 \). State the other factor(s).

c) Use the results from part a) to decide whether \( x + y \) or \( x - y \) is a factor of \( x^3 - y^3 \). State the other factor(s).

d) Use your findings to factor \( x^6 + y^6 \).

C1 Explain to a classmate how to use the graph of \( f(x) = x^4 - 3x^2 - 4 \) to determine at least one binomial factor of the polynomial. What are all of the factors of the polynomial?

C2 Identify the possible factors of the expression \( x^4 - x^3 + 2x^2 - 5 \). Explain your reasoning in more than one way.

C3 How can the factor theorem, the integral zero theorem, the quadratic formula, and synthetic division be used together to factor a polynomial of degree greater than or equal to three?