2.1 Radical Functions and Transformations

Focus on . . .
- investigating the function \( y = \sqrt{x} \) using a table of values and a graph
- graphing radical functions using transformations
- identifying the domain and range of radical functions

Does a feather fall more slowly than a rock? Galileo Galilei, a mathematician and scientist, pondered this question more than 400 years ago. He theorized that the rate of falling objects depends on air resistance, not on mass. It is believed that he tested his idea by dropping spheres of different masses but the same diameter from the top of the Leaning Tower of Pisa in what is now Italy. The result was exactly as he predicted—they fell at the same rate.

In 1971, during the Apollo 15 lunar landing, Commander David Scott performed a similar demonstration on live television. Because the surface of the moon is essentially a vacuum, a hammer and a feather fell at the same rate.

For objects falling near the surface of Earth, the function \( d = 5t^2 \) approximately models the time, \( t \), in seconds, for an object to fall a distance, \( d \), in metres, if the resistance caused by air can be ignored.

1. a) Identify any restrictions on the domain of this function. Why are these restrictions necessary? What is the range of the function?

   b) Create a table of values and a graph showing the distance fallen as a function of time.

2. Express time in terms of distance for the distance-time function from step 1. Represent the new function graphically and using a table of values.

3. For each representation, how is the equation of the new function from step 2 related to the original function?
Reflect and Respond

4. a) The original function is a distance-time function. What would you call the new function? Under what circumstances would you use each function?

b) What is the shape of the graph of the original function? Describe the shape of the graph of the new function.

Link the Ideas

The function that gives the predicted fall time for an object under the influence of gravity is an example of a radical function. Radical functions have restricted domains if the index of the radical is an even number. Like many types of functions, you can represent radical functions in a variety of ways, including tables, graphs, and equations. You can create graphs of radical functions using tables of values or technology, or by transforming the base radical function, \( y = \sqrt{x} \).

Example 1

Graph Radical Functions Using Tables of Values

Use a table of values to sketch the graph of each function. Then, state the domain and range of each function.

a) \( y = \sqrt{x} \)  
   b) \( y = \sqrt{x - 2} \)  
   c) \( y = \sqrt{x - 3} \)

Solution

a) For the function \( y = \sqrt{x} \), the radicand \( x \) must be greater than or equal to zero, \( x \geq 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
</tr>
</tbody>
</table>

How can you choose values of \( x \) that allow you to complete the table without using a calculator?

The graph has an endpoint at \( (0, 0) \) and continues up and to the right. The domain is \( \{x \mid x \geq 0, x \in \mathbb{R}\} \). The range is \( \{y \mid y \geq 0, y \in \mathbb{R}\} \).
b) For the function \( y = \sqrt{x - 2} \), the value of the radicand must be greater than or equal to zero.

\[
x - 2 \geq 0 \\
x \geq 2
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
</tr>
<tr>
<td>27</td>
<td>5</td>
</tr>
</tbody>
</table>

How is this table related to the table for \( y = \sqrt{x} \) in part a)?

How does the graph of \( y = \sqrt{x - 2} \) compare to the graph of \( y = \sqrt{x} \)?

The domain is \( \{x \mid x \geq 2, x \in \mathbb{R}\} \). The range is \( \{y \mid y \geq 0, y \in \mathbb{R}\} \).

c) The radicand of \( y = \sqrt{x - 3} \) must be non-negative.

\( x \geq 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>2</td>
</tr>
</tbody>
</table>

How does the graph of \( y = \sqrt{x - 3} \) compare to the graph of \( y = \sqrt{x} \)?

The domain is \( \{x \mid x \geq 0, x \in \mathbb{R}\} \) and the range is \( \{y \mid y \geq -3, y \in \mathbb{R}\} \).

**Your Turn**

Sketch the graph of the function \( y = \sqrt{x + 5} \) using a table of values. State the domain and the range.
Graphing Radical Functions Using Transformations

You can graph a radical function of the form \( y = a\sqrt{b(x - h)} + k \) by transforming the graph of \( y = \sqrt{x} \) based on the values of \( a, b, h, \) and \( k \). The effects of changing parameters in radical functions are the same as the effects of changing parameters in other types of functions.

- Parameter \( a \) results in a vertical stretch of the graph of \( y = \sqrt{x} \) by a factor of \( |a| \). If \( a < 0 \), the graph of \( y = \sqrt{x} \) is reflected in the \( x \)-axis.
- Parameter \( b \) results in a horizontal stretch of the graph of \( y = \sqrt{x} \) by a factor of \( \frac{1}{|b|} \). If \( b < 0 \), the graph of \( y = \sqrt{x} \) is reflected in the \( y \)-axis.
- Parameter \( h \) determines the horizontal translation. If \( h > 0 \), the graph of \( y = \sqrt{x} \) is translated to the right \( h \) units. If \( h < 0 \), the graph is translated to the left \( |h| \) units.
- Parameter \( k \) determines the vertical translation. If \( k > 0 \), the graph of \( y = \sqrt{x} \) is translated up \( k \) units. If \( k < 0 \), the graph is translated down \( |k| \) units.

Example 2

Graph Radical Functions Using Transformations

Sketch the graph of each function using transformations. Compare the domain and range to those of \( y = \sqrt{x} \) and identify any changes.

a) \( y = 3\sqrt{-(x - 1)} \)

b) \( y - 3 = -\sqrt{2x} \)

Solution

a) The function \( y = 3\sqrt{-(x - 1)} \) is expressed in the form \( y = a\sqrt{b(x - h)} + k \). Identify the value of each parameter and how it will transform the graph of \( y = \sqrt{x} \).
- \( a = 3 \) results in a vertical stretch by a factor of 3 (step 1).
- \( b = -1 \) results in a reflection in the \( y \)-axis (step 2).
- \( h = 1 \) results in a horizontal translation of 1 unit to the right (step 3).
- \( k = 0 \), so the graph has no vertical translation.

Method 1: Transform the Graph Directly

Start with a sketch of \( y = \sqrt{x} \) and apply the transformations one at a time.
Method 2: Map Individual Points

Choose key points on the graph of \( y = \sqrt{x} \) and map them for each transformation.

<table>
<thead>
<tr>
<th>Transformation of ( y = \sqrt{x} )</th>
<th>Mapping</th>
</tr>
</thead>
</table>
| Vertical stretch by a factor of 3   | (0, 0) → (0, 0)  
(1, 1) → (1, 3)  
(4, 2) → (4, 6)  
(9, 3) → (9, 9) |
| Horizontal reflection in the \( y \)-axis | (0, 0) → (0, 0)  
(1, 3) → (−1, 3)  
(4, 6) → (−4, 6)  
(9, 9) → (−9, 9) |
| Horizontal translation of 1 unit to the right | (0, 0) → (1, 0)  
(−1, 3) → (0, 3)  
(−4, 6) → (−3, 6)  
(−9, 9) → (−8, 9) |

Plot the image points to create the transformed graph.

The function \( y = \sqrt{x} \) is reflected horizontally, stretched vertically by a factor of 3, and then translated 1 unit right. So, the graph of \( y = 3\sqrt{(x - 1)} \) extends to the left from \( x = 1 \) and its domain is \( \{ x \mid x \leq 1, x \in \mathbb{R} \} \).

Since the function is not reflected vertically or translated vertically, the graph of \( y = 3\sqrt{(x - 1)} \) extends up from \( y = 0 \), similar to the graph of \( y = \sqrt{x} \). The range, \( \{ y \mid y \geq 0, y \in \mathbb{R} \} \), is unchanged by the transformations.

b) Express the function \( y - 3 = -\sqrt{2x} \) in the form \( y = a\sqrt{b(x - h)} + k \) to identify the value of each parameter.

\[
y - 3 = -\sqrt{2x} \\
y = -\sqrt{2x} + 3
\]

- \( b = 2 \) results in horizontal stretch by a factor of \( \frac{1}{2} \) (step 1).
- \( a = -1 \) results in a reflection in the \( x \)-axis (step 2).
- \( h = 0 \), so the graph is not translated horizontally.
- \( k = 3 \) results in a vertical translation of 3 units up (step 3).

Apply these transformations either directly to the graph of \( y = \sqrt{x} \) or to key points, and then sketch the transformed graph.
Method 1: Transform the Graph Directly
Use a sketch of \( y = \sqrt{x} \) and apply the transformations to the curve one at a time.

Method 2: Use Mapping Notation
Apply each transformation to the point \((x, y)\) to determine a general mapping notation for the transformed function.

<table>
<thead>
<tr>
<th>Transformation of ( y = \sqrt{x} )</th>
<th>Mapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal stretch by a factor of ( \frac{1}{2} )</td>
<td>((x, y) \rightarrow \left( \frac{1}{2} x, y \right))</td>
</tr>
<tr>
<td>Reflection in the x-axis</td>
<td>(\left( \frac{1}{2} x, y \right) \rightarrow \left( \frac{1}{2} x, -y \right))</td>
</tr>
<tr>
<td>Vertical translation of 3 units up</td>
<td>(\left( \frac{1}{2} x, -y \right) \rightarrow \left( \frac{1}{2} x, -y + 3 \right))</td>
</tr>
</tbody>
</table>

Choose key points on the graph of \( y = \sqrt{x} \) and use the general mapping notation \((x, y) \rightarrow \left( \frac{1}{2} x, -y + 3 \right)\) to determine their image points on the function \( y - 3 = -\sqrt{2x} \).

\[(0, 0) \rightarrow (0, 3)\]
\[(1, 1) \rightarrow (0.5, 2)\]
\[(4, 2) \rightarrow (2, 1)\]
\[(9, 3) \rightarrow (4.5, 0)\]

Since there are no horizontal reflections or translations, the graph still extends to the right from \( x = 0 \). The domain, \( \{x \mid x \geq 0, x \in \mathbb{R}\} \), is unchanged by the transformations as compared with \( y = \sqrt{x} \).

The function is reflected vertically and then translated 3 units up, so the graph extends down from \( y = 3 \). The range is \( \{y \mid y \leq 3, y \in \mathbb{R}\} \), which has changed as compared to \( y = \sqrt{x} \).
Your Turn

a) Sketch the graph of the function \( y = -2\sqrt{x + 3} - 1 \) by transforming the graph of \( y = \sqrt{x} \).
b) Identify the domain and range of \( y = \sqrt{x} \) and describe how they are affected by the transformations.

Example 3

Determine a Radical Function From a Graph

Mayleen is designing a symmetrical pattern. She sketches the curve shown and wants to determine its equation and the equation of its reflection in each quadrant. The graph is a transformation of the graph of \( y = \sqrt{x} \). What are the equations of the four functions Mayleen needs to work with?

Solution

The base function \( y = \sqrt{x} \) is not reflected or translated, but it is stretched. A radical function that involves a stretch can be obtained from either a vertical stretch or a horizontal stretch. Use an equation of the form \( y = a\sqrt{x} \) or \( y = \sqrt{bx} \) to represent the image function for each type of stretch.

Method 1: Compare Vertical or Horizontal Distances

Superimpose the graph of \( y = \sqrt{x} \) and compare corresponding distances to determine the factor by which the function has been stretched.

View as a Vertical Stretch \( (y = a\sqrt{x}) \)
Each vertical distance is 2 times the corresponding distance for \( y = \sqrt{x} \).

View as a Horizontal Stretch \( (y = \sqrt{bx}) \)
Each horizontal distance is \( \frac{1}{4} \) of the corresponding distance for \( y = \sqrt{x} \).

This represents a vertical stretch by a factor of 2, which means \( a = 2 \). The equation \( y = 2\sqrt{x} \) represents the function.

This represents a horizontal stretch by a factor of \( \frac{1}{4} \), which means \( b = 4 \). The equation \( y = \sqrt{4x} \) represents the function.

Express the equation of the function as either \( y = 2\sqrt{x} \) or \( y = \sqrt{4x} \).
Method 2: Substitute Coordinates of a Point

Use the coordinates of one point on the function, such as \((1, 2)\), to determine the stretch factor.

**View as a Vertical Stretch**

Substitute 1 for \(x\) and 2 for \(y\) in the equation \(y = a\sqrt{x}\). Then, solve for \(a\).

\[
\begin{align*}
y &= a\sqrt{x} \\
2 &= a\sqrt{1} \\
2 &= a \\
The \text{equation of the function is } y &= 2\sqrt{x}.
\end{align*}
\]

**View as a Horizontal Stretch**

Substitute the coordinates \((1, 2)\) in the equation \(y = \sqrt{bx}\) and solve for \(b\).

\[
\begin{align*}
y &= \sqrt{bx} \\
2 &= \sqrt{b(1)} \\
2 &= \sqrt{b} \\
2^2 &= (\sqrt{b})^2 \\
4 &= b
\end{align*}
\]

The equation can also be expressed as \(y = \sqrt{4x}\).

Represent the function in simplest form by \(y = 2\sqrt{x}\) or by \(y = \sqrt{4x}\).

Determine the equations of the reflected curves using \(y = 2\sqrt{x}\).

- A reflection in the \(y\)-axis results in the function \(y = 2\sqrt{-x}\), since \(b = -1\).
- A reflection in the \(x\)-axis results in \(y = -2\sqrt{x}\), since \(a = -1\).

Reflecting these graphs into the third quadrant results in the function \(y = -2\sqrt{-x}\).

Mayleen needs to use the equations \(y = 2\sqrt{x}\), \(y = 2\sqrt{-x}\), \(y = -2\sqrt{x}\), and \(y = -2\sqrt{-x}\). Similarly, she could use the equations \(y = \sqrt{4x}\), \(y = \sqrt{-4x}\), \(y = -\sqrt{4x}\), and \(y = -\sqrt{-4x}\).

**Your Turn**

**a)** Determine two forms of the equation for the function shown.

The function is a transformation of the function \(y = \sqrt{x}\).

**b)** Show algebraically that the two equations are equivalent.

**c)** What is the equation of the curve reflected in each quadrant?
Model the Speed of Sound

Justin’s physics textbook states that the speed, \( s \), in metres per second, of sound in dry air is related to the air temperature, \( T \), in degrees Celsius, by the function \( s = 331.3 \sqrt{1 + \frac{T}{273.15}} \).

a) Determine the domain and range in this context.

b) On the Internet, Justin finds another formula for the speed of sound, \( s = 20\sqrt{T + 273} \). Use algebra to show that the two functions are approximately equivalent.

c) How is the graph of this function related to the graph of the base square root function? Which transformation do you predict will be the most noticeable on a graph?

d) Graph the function \( s = 331.3 \sqrt{1 + \frac{T}{273.15}} \) using technology.

e) Determine the speed of sound, to the nearest metre per second, at each of the following temperatures.

i) 20 °C (normal room temperature)

ii) 0 °C (freezing point of water)

iii) −63 °C (coldest temperature ever recorded in Canada)

iv) −89 °C (coldest temperature ever recorded on Earth)

Solution

a) Use the following inequality to determine the domain:

\[
1 + \frac{T}{273.15} \geq 0
\]

\[
\frac{T}{273.15} \geq -1
\]

\[
T \geq -273.15
\]

The domain is \( \{ T \mid T \geq -273.15, T \in \mathbb{R} \} \). This means that the temperature must be greater than or equal to −273.15 °C, which is the lowest temperature possible and is referred to as absolute zero.

The range is \( \{ s \mid s \geq 0, s \in \mathbb{R} \} \), which means that the speed of sound is a non-negative value.

b) Rewrite the function from the textbook in simplest form.

\[
s = 331.3 \sqrt{1 + \frac{T}{273.15}}
\]

\[
s = 331.3 \sqrt{\frac{273.15}{273.15} + \frac{T}{273.15}}
\]

\[
s = 331.3 \sqrt{\frac{273.15 + T}{273.15}}
\]

\[
s = 331.3 \frac{\sqrt{273.15 + T}}{\sqrt{273.15}}
\]

\[
s \approx 20\sqrt{T + 273}
\]

How could you verify that these expressions are approximately equivalent?

The function found on the Internet, \( s = 20\sqrt{T + 273} \), is the approximate simplest form of the function in the textbook.
c) Analyse the transformations and determine the order in which they must be performed.

The graph of \( s = \sqrt{T} \) is stretched vertically by a factor of about 20 and then translated about 273 units to the left. Translating 273 units to the left will be most noticeable on the graph of the function.


d) 

![Graph of the function](image)

Are these transformations consistent with the domain and range?

Are your answers to part c) confirmed by the graph?

e) 

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Approximate Speed of Sound (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) 20</td>
<td>343</td>
</tr>
<tr>
<td>ii) 0</td>
<td>331</td>
</tr>
<tr>
<td>iii) -63</td>
<td>291</td>
</tr>
<tr>
<td>iv) -89</td>
<td>272</td>
</tr>
</tbody>
</table>

Your Turn

A company estimates its cost of production using the function \( C(n) = 20\sqrt{n} + 1000 \), where \( C \) represents the cost, in dollars, to produce \( n \) items.

a) Describe the transformations represented by this function as compared to \( C = \sqrt{n} \).

b) Graph the function using technology. What does the shape of the graph imply about the situation?

c) Interpret the domain and range in this context.

d) Use the graph to determine the expected cost to produce 12,000 items.

Did You Know?

Eureka, on Ellesmere Island, Nunavut, holds the North American record for the lowest-ever average monthly temperature, \(-47.9 °C\) in February 1979. For 18 days, the temperature stayed below \(-45 °C\).
Key Ideas

- The base radical function is $y = \sqrt{x}$. Its graph has the following characteristics:
  - a left endpoint at (0, 0)
  - no right endpoint
  - the shape of half of a parabola
  - a domain of $\{x | x \geq 0, x \in \mathbb{R}\}$ and a range of $\{y | y \geq 0, y \in \mathbb{R}\}$

- You can graph radical functions of the form $y = a\sqrt{b(x - h)} + k$ by transforming the base function $y = \sqrt{x}$.

- You can analyse transformations to identify the domain and range of a radical function of the form $y = a\sqrt{b(x - h)} + k$.

Check Your Understanding

Practise

1. Graph each function using a table of values. Then, identify the domain and range.
   a) $y = \sqrt{x} - 1$
   b) $y = \sqrt{x} + 6$
   c) $y = \sqrt{3 - x}$
   d) $y = \sqrt{-2x - 5}$

2. Explain how to transform the graph of $y = \sqrt{x}$ to obtain the graph of each function. State the domain and range in each case.
   a) $y = 7\sqrt{x} - 9$
   b) $y = \sqrt{-x} + 8$
   c) $y = -\sqrt{0.2x}$
   d) $4 + y = \frac{1}{3}\sqrt{x} + 6$

3. Match each function with its graph.
   a) $y = \sqrt{x} - 2$
   b) $y = \sqrt{-x} + 2$
   c) $y = -\sqrt{x} + 2$
   d) $y = -\sqrt{-x - 2}$
4. Write the equation of the radical function that results by applying each set of transformations to the graph of $y = \sqrt{x}$.

a) vertical stretch by a factor of 4, then horizontal translation of 6 units left
b) horizontal stretch by a factor of $\frac{1}{8}$, then vertical translation of 5 units down
c) horizontal reflection in the $y$-axis, then horizontal translation of 4 units right and vertical translation of 11 units up
d) vertical stretch by a factor of 0.25, vertical reflection in the $x$-axis, and horizontal stretch by a factor of 10

5. Sketch the graph of each function using transformations. State the domain and range of each function.

a) $f(x) = \sqrt{-x} - 3$
b) $r(x) = 3\sqrt{x} + 1$
c) $p(x) = -\sqrt{x} - 2$
d) $y - 1 = -\sqrt{-4(x - 2)}$
e) $m(x) = \sqrt{\frac{1}{2}x + 4}$
f) $y + 1 = \frac{1}{3}\sqrt{-(x + 2)}$

**Apply**

6. Consider the function $f(x) = \frac{1}{4}\sqrt{5x}$.

a) Identify the transformations represented by $f(x)$ as compared to $y = \sqrt{x}$.
b) Write two functions equivalent to $f(x)$: one of the form $y = a\sqrt{x}$ and the other of the form $y = \sqrt{bx}$
c) Identify the transformation(s) represented by each function you wrote in part b).
d) Use transformations to graph all three functions. How do the graphs compare?

7. a) Express the radius of a circle as a function of its area.
b) Create a table of values and a graph to illustrate the relationship that this radical function represents.

8. For an observer at a height of $h$ feet above the surface of Earth, the approximate distance, $d$, in miles, to the horizon can be modelled using the radical function $d = \sqrt{1.50h}$.

a) Use the language of transformations to describe how to obtain the graph from the base square root graph.
b) Determine an approximate equivalent function of the form $d = a\sqrt{h}$ for the function. Which form of the function do you prefer, and why?
c) A lifeguard on a tower is looking out over the water with binoculars. How far can she see if her eyes are 20 ft above the level of the water? Express your answer to the nearest tenth of a mile.

9. The function $4 - y = \sqrt{3x}$ is translated 9 units up and reflected in the $x$-axis.

a) Without graphing, determine the domain and range of the image function.
b) Compared to the base function, $y = \sqrt{x}$, by how many units and in which direction has the given function been translated horizontally? vertically?
10. For each graph, write the equation of a radical function of the form \( y = a \sqrt{b(x - h)} + k \).

\[ \text{a) } \]
\[ \text{b) } \]
\[ \text{c) } \]
\[ \text{d) } \]

11. Write the equation of a radical function with each domain and range.

\[ \text{a) } \{x \mid x \geq 6, x \in \mathbb{R}, y \mid y \geq 1, y \in \mathbb{R} \} \]
\[ \text{b) } \{x \mid x \geq -7, x \in \mathbb{R}, y \mid y \leq -9, y \in \mathbb{R} \} \]
\[ \text{c) } \{x \mid x \leq 4, x \in \mathbb{R}, y \mid y \geq -3, y \in \mathbb{R} \} \]
\[ \text{d) } \{x \mid x \leq -5, x \in \mathbb{R}, y \mid y \leq 8, y \in \mathbb{R} \} \]

12. Agronomists use radical functions to model and optimize corn production. One factor they analyse is how the amount of nitrogen fertilizer applied affects the crop yield. Suppose the function \( Y(n) = 760 \sqrt{n} + 2000 \) is used to predict the yield, \( Y \), in kilograms per hectare, of corn as a function of the amount, \( n \), in kilograms per hectare, of nitrogen applied to the crop.

\[ \text{a) } \]
\[ \text{b) } \]
\[ \text{c) } \]
\[ \text{d) } \]

Did You Know?

Over 6300 years ago, the Indigenous people in the area of what is now Mexico domesticated and cultivated several varieties of corn. The cultivation of corn is now global.
13. A manufacturer wants to predict the consumer interest in a new smart phone. The company uses the function \( P(d) = -2\sqrt{-d} + 20 \) to model the number, \( P \), in millions, of pre-orders for the phone as a function of the number, \( d \), of days before the phone’s release date.

a) What are the domain and range and what do they mean in this situation?

b) Identify the transformations represented by the function as compared to \( y = \sqrt{d} \).

c) Graph the function and explain what the shape of the graph indicates about the situation.

d) Determine the number of pre-orders the manufacturer can expect to have 30 days before the release date.

14. During election campaigns, campaign managers use surveys and polls to make projections about the election results. One campaign manager uses a radical function to model the possible error in polling predictions as a function of the number of days until the election, as shown in the graph.

a) Explain what the graph shows about the accuracy of polls before elections.

b) Determine an equation to represent the function. Show how you developed your answer.

c) Describe the transformations that the function represents as compared to \( y = \sqrt{x} \).

15. While meeting with a client, a manufacturer of custom greenhouses sketches a greenhouse in the shape of the graph of a radical function. What equation could the manufacturer use to represent the shape of the greenhouse roof?

16. Determine the equation of a radical function with

a) endpoint at (2, 5) and passing through the point (6, 1)

b) endpoint at (3, –2) and an x-intercept with a value of –6
17. The Penrose method is a system for giving voting powers to members of assemblies or legislatures based on the square root of the number of people that each member represents, divided by 1000. Consider a parliament that represents the people of the world and how voting power might be given to different nations. The table shows the estimated populations of Canada and the three most populous and the three least populous countries in the world.

<table>
<thead>
<tr>
<th>Country</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>1 361 513 000</td>
</tr>
<tr>
<td>India</td>
<td>1 251 696 000</td>
</tr>
<tr>
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</tr>
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</table>

a) Share your answers to the following two questions with a classmate and explain your thinking:
   • Which countries might feel that a “one nation, one vote” system is an unfair way to allocate voting power?
   • Which countries might feel that a “one person, one vote” system is unfair?

b) What percent of the voting power would each nation listed above have under a “one person, one vote” system, assuming a world population of approximately 7.302 billion?

c) If \( x \) represents the population of a country and \( V(x) \) represents its voting power, what function could be written to represent the Penrose method?

d) Under the Penrose method, the sum of the world voting power using the given data is approximately 765. What percent of the voting power would this system give each nation in the table?

e) Why might the Penrose method be viewed as a compromise for allocating voting power?

18. **MINI LAB** The period of a pendulum is the time for one complete swing back and forth. As long as the initial swing angle is kept relatively small, the period of a pendulum is related to its length by a radical function.

![Diagram of a pendulum with a length of 30 cm]

**Materials**
- thread
- washer or other suitable mass
- tape
- ruler
- stopwatch or timer

**Step 1** Tie a length of thread to a washer or other mass. Tape the thread to the edge of a table or desk top so that the length between the pivot point and the centre of the washer is 30 cm.

**Step 2** Pull the mass to one side and allow it to swing freely. Measure the total time for 10 complete swings back and forth and then divide by 10 to determine the period for this length. Record the length and period in a table.

**Step 3** Repeat steps 1 and 2 using lengths of 25 cm, 20 cm, 15 cm, 10 cm, 5 cm, and 3 cm (and shorter distances if possible).

**Step 4** Create a scatter plot showing period as a function of length. Draw a smooth curve through or near the points. Does it appear to be a radical function? Justify your answer.

**Step 5** What approximate transformation(s) to the graph of \( y = \sqrt{x} \) would produce your result? Write a radical function that approximates the graph, where \( T \) represents the period and \( L \) represents the length of the pendulum.
**Extend**

19. The inverse of \( f(x) = \sqrt{x} \) is 
\[ f^{-1}(x) = x^2, \; x \geq 0. \]

a) Graph both functions, and use them to explain why the restriction is necessary on the domain of the inverse function.

b) Determine the equation, including any restrictions, of the inverse of each of the following functions.

i) \( g(x) = -\sqrt{x - 5} \)

ii) \( h(x) = \sqrt{-x} + 3 \)

iii) \( j(x) = \sqrt{2x - 7} - 6 \)

20. If \( f(x) = \frac{5}{8} \sqrt{-\frac{7}{12} x} \) and 
\[ g(x) = -\frac{2}{5} \sqrt{6(x + 3)} - 4, \] what transformations could you apply to the graph of \( f(x) \) to create the graph of \( g(x) \)?

**Create Connections**

C1 Which parameters in \( y = a \sqrt{b(x - h)} + k \) affect the domain of \( y = \sqrt{x} \)? Which parameters affect the range? Explain, using examples.

C2 Sarah claims that any given radical function can be simplified so that there is no value of \( b \), only a value of \( a \). Is she correct? Explain, using examples.

C3 Compare and contrast the process of graphing a radical function using transformations with graphing a quadratic function using transformations.

C4 **Mini Lab** The Wheel of Theodorus, or Square Root Spiral, is a geometric construction that contains line segments with length equal to the square root of any whole number.

**Materials**
- ruler, drafting square, or other object with a right angle
- millimetre ruler

Step 1 Create an isosceles right triangle with legs that are each 1 cm long. Mark one end of the hypotenuse as point C. What is the length of the hypotenuse, expressed as a radical?

Step 2 Use the hypotenuse of the first triangle as one leg of a new right triangle. Draw a length of 1 cm as the other leg, opposite point C. What is the length of the hypotenuse of this second triangle, expressed as a radical?

Step 3 Continue to create right triangles, each time using the hypotenuse of the previous triangle as a leg of the next triangle, and a length of 1 cm as the other leg (drawn so that the 1-cm leg is opposite point C). Continue the spiral until you would overlap the initial base.

Step 4 Create a table to represent the length of the hypotenuse as a function of the triangle number (first, second, third triangle in the pattern, etc.). Express lengths both in exact radical form and in approximate decimal form.

Step 5 Write an equation to represent this function, where \( L \) represents the hypotenuse length and \( n \) represents the triangle number. Does the equation involve any transformations on the base square root function? Explain.